

# Properties of $G$ -martingales with finite variation and the application to $G$ -Sobolev spaces

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## Abstract

As is known, a process of form  $\int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)ds$ ,  $\eta \in M_G^1(0, T)$ , is a non-increasing  $G$ -martingale. In this paper, we shall show that a non-increasing  $G$ -martingale could not be form of  $\int_0^t \eta_s ds$  or  $\int_0^t \gamma_s d\langle B \rangle_s$ ,  $\eta, \gamma \in M_G^1(0, T)$ , which implies that the decomposition for generalized  $G$ -Itô processes is unique: For  $\zeta \in H_G^1(0, T)$ ,  $\eta \in M_G^1(0, T)$  and non-increasing  $G$ -martingales  $K, L$ , if

$$\int_0^t \zeta_s dB_s + \int_0^t \eta_s ds + K_t = L_t, \quad t \in [0, T],$$

then we have  $\eta \equiv 0$ ,  $\zeta \equiv 0$  and  $K_t = L_t$ . As an application, we give a characterization to the  $G$ -Sobolev spaces introduced in Peng and Song (2015).

**Key words:**  $G$ -martingales with finite variation; generalized  $G$ -Itô processes; unique decomposition;  $G$ -Sobolev spaces

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## 1 Introduction

The notion of  $G$ -expectation is a type of nonlinear expectation proposed by Peng [3, 6]. It can be regarded as a nonlinear generalization of Wiener probability space  $(\Omega, \mathcal{F}, P)$  where  $\Omega = C_0([0, \infty), \mathbb{R}^d)$  equipped with the uniform norm,  $\mathcal{F} = \mathcal{B}(\Omega)$  and  $P$  is a Wiener probability measure defined on  $(\Omega, \mathcal{F})$ . Recall that the Wiener measure is defined such that the canonical process  $B_t(\omega) := \omega_t$ ,  $t \geq 0$  is a continuous process with stationary and independent increments, namely  $(B_t)_{t \geq 0}$  is a Brownian motion.  $G$ -expectation  $\mathbb{E}$  is a sublinear expectation on the same canonical space  $\Omega$ , such that the same canonical process  $B$  is a  $G$ -Brownian motion, i.e., it is a continuous process with stationary and independent increments. A crucial difference is that the quadratic variance process  $\langle B \rangle$  of the  $G$ -Brownian motion  $B$  is no longer a deterministic function of the time variable  $t$ . It is a process with stationary and independent increments. For the one-dimensional case, its increments are bounded by  $\bar{\sigma}^2 := \mathbb{E}[B_1^2] \geq -\mathbb{E}[-B_1^2] =: \underline{\sigma}^2$ ,

$$\underline{\sigma}^2(t-s) \leq \langle B \rangle_t - \langle B \rangle_s \leq \bar{\sigma}^2(t-s), \quad \text{for } s < t. \quad (1.1)$$

Similar to the classical Brownian motion, the  $G$ -Brownian motion corresponds to a (fully nonlinear) PDE: For a function  $\varphi \in C_{b,Lip}(\mathbb{R})$ , the collection of bounded Lipschitz functions

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on  $\mathbb{R}$ , the function  $u(t, x) := \mathbb{E}[\varphi(x + B_t)]$  is the (viscosity) solution to the following  $G$ -heat equation

$$\begin{aligned}\partial_t u - G(\partial_x^2 u) &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) &= \varphi(x),\end{aligned}$$

where  $G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$ ,  $a \in \mathbb{R}$ . Moreover, for fixed  $T > 0$ , the process  $u(T - t, B_t)$ ,  $t \in [0, T]$  is a martingale under  $G$ -expectation. By Itô's formula, one has

$$\begin{aligned}u(T - t, B_t) &= \mathbb{E}[\varphi(B_T)] + \int_0^t \partial_x u(T - s, B_s) dB_s \\ &\quad + \frac{1}{2} \int_0^t \partial_x^2 u(T - s, B_s) d\langle B \rangle_s - \int_0^t G(\partial_x^2 u)(T - s, B_s) ds.\end{aligned}$$

The process  $M_t := \int_0^t \partial_x u(T - s, B_s) dB_s$  is a symmetric  $G$ -martingale (i.e.,  $M$  and  $-M$  are both  $G$ -martingales), which shares the same properties with classical martingales in the probability space. The process  $K_t := \frac{1}{2} \int_0^t \partial_x^2 u(T - s, B_s) d\langle B \rangle_s - \int_0^t G(\partial_x^2 u)(T - s, B_s) ds$  is a non-increasing  $G$ -martingale. For the linear case ( $\underline{\sigma} = \bar{\sigma}$ ), this term disappears. However, when  $\underline{\sigma} < \bar{\sigma}$ ,  $G$ -martingales with finite variation are a class of nontrivial processes, which show the variance uncertainty of  $G$ -expectation.

For  $Z \in H_G^2(0, T)$ ,  $\eta \in M_G^2(0, T)$ , [4] showed that a process of form

$$X_t = X_0 + \int_0^t Z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds \quad (1.2)$$

is a  $G$ -martingale, and conjectured that for any  $\xi \in L_G^2(\Omega_T)$ , the martingale  $\mathbb{E}_t[\xi]$  has the representation (1.2). [4] proved this conjecture for cylinder random variables of form  $\xi = \varphi(B_{t_1}, \dots, B_{t_n})$ . For the general case, Soner et al (2011) and Song (2011) proved independently the following  $G$ -martingale decomposition theorem:

$$\mathbb{E}_t[\xi] = \mathbb{E}[\xi] + \int_0^t Z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s + K_t,$$

where  $K_t$  is a non-increasing  $G$ -martingale.

In this paper, our interest concentrates on  $G$ -martingales with finite variation. In the  $G$ -expectation space, there are three types of processes whose variation is finite.

- (1)  $L_t = \int_0^t \eta_s ds$ ,  $\eta \in M_G^p(0, T)$ ;
- (2)  $A_t = \int_0^t \zeta_s d\langle B \rangle_s$ ,  $\zeta \in M_G^p(0, T)$ ;
- (3)  $G$ -martingales with finite variation.

It is a very important problem to distinguish these three types of processes. Song (2012) distinguished (1) and (2) completely:

$$\int_0^t \eta_s ds = \int_0^t \zeta_s d\langle B \rangle_s, \quad t \in [0, T] \implies \eta \equiv \zeta = 0. \quad (1.3)$$

As an immediate corollary of this result, Song (2012) proved the uniqueness of the representation for  $G$ -martingales with finite variation. Also, Conclusion (1.3) implies that the decomposition of  $G$ -Itô process is unique, which is crucial for Peng and Song (2015) to define the  $G$ -Sobolev space  $W_G^{1,2;p}(0, T)$ .

The main job of this paper is to distinguish  $G$ -martingales with finite variation from the other two types of processes. For a  $G$ -martingale of the form  $K_t(\varsigma) = \int_0^t \varsigma_s d\langle B \rangle_s - \int_0^t 2G(\varsigma_s)ds$ , if  $K_t(\varsigma) = \int_0^t \eta_s ds$  (resp.  $\int_0^t \zeta_s d\langle B \rangle_s$ ),  $t \in [0, T]$ , then by Conclusion (1.3), we get  $\varsigma \equiv \eta = 0$  (resp.  $\varsigma \equiv \zeta = 0$ ). So a  $G$ -martingale  $K_t(\varsigma)$  could not be form of (1) or (2). Here we shall prove this conclusion for general  $G$ -martingales:

A  $G$ -martingale with finite variation could not be form of  $\int_0^t \eta_s ds$  or  $\int_0^t \zeta_s d\langle B \rangle_s$ .

More precisely, let  $K$  be a non-increasing  $G$ -martingale. If

$$K_t = \int_0^t \eta_s ds \text{ (resp. } \int_0^t \zeta_s d\langle B \rangle_s), \quad t \in [0, T],$$

we conclude that  $K \equiv 0$ .

Based on this conclusion, we can prove that the decomposition for generalized  $G$ -Itô processes is unique: For  $\zeta \in H_G^1(0, T)$ ,  $\eta \in M_G^1(0, T)$  and non-increasing  $G$ -martingales  $K, L$ , if

$$\int_0^t \zeta_s dB_s + \int_0^t \eta_s ds + K_t = L_t, \quad t \in [0, T],$$

then we have  $\eta \equiv 0$ ,  $\zeta \equiv 0$  and  $K_t = L_t$ . This turns out to be a very strong result. Many important conclusions in the context of  $G$ -expectation theory, including Conclusion (1.3), can be considered as its immediate corollaries (see Remark 3.12 for details). The main results of this paper are Theorem 3.6 and Theorem 3.10.

Peng and Song (2015) introduced the notion of  $G$ -Sobolev spaces. In the  $G$ -Sobolev space  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$  the authors defined solutions to the following path dependent PDEs:

$$\begin{aligned} \mathcal{D}_t u + G(\mathcal{D}_x^2 u) + f(t, u, \mathcal{D}_x u) &= 0, \quad t \in [0, T], \\ u_T &= \xi. \end{aligned} \tag{1.4}$$

This  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}$ -solution corresponds to the solution of the backward SDEs driven by  $G$ -Brownian motion considered in Hu et al (2014).

In this paper, as an application of the main results, we shall give a characterization of the  $G$ -Sobolev space  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$ . The main idea is, just like the liner case, to integrate  $\mathcal{A}_G u = \mathcal{D}_t u + G(\mathcal{D}_x^2 u)$  as one operator, which reduces the regularity requirement for the solutions. To well define the derivative  $\mathcal{A}_G u$  for  $u \in W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$ , the uniqueness of the decomposition for generalized  $G$ -Itô processes plays a crucial role.

The rest of the paper is organized as follows. In Section 2, we present some basic notions and definitions on the  $G$ -expectation theory. We shall prove the main results in Section 3. As an application of the uniqueness of the decomposition for generalized  $G$ -Itô processes, we shall refine the definition of the  $G$ -Sobolev space  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}$  in Section 4. In Section 5, as an appendix, we present the wellposedness result of  $G$ -BSDEs obtained in [2].

## 2 Some definitions and notations about $G$ -expectation

We review some basic notions and definitions on the  $G$ -expectation theory. The readers may refer to [3], [4], [5], [6] for more details.

Let  $\Omega_T = C_0([0, T]; \mathbb{R}^d)$  be the space of all  $\mathbb{R}^d$ -valued continuous paths  $\omega = (\omega(t))_{t \in [0, T]}$  with  $\omega(0) = 0$  and let  $B_t(\omega) = \omega(t)$  be the canonical process.

Let us recall the definitions of  $G$ -Brownian motion and its corresponding  $G$ -expectation introduced in [4]. For simplicity, here we only consider the one-dimensional case.

Set

$$L_{ip}(\Omega_T) := \{\varphi(\omega(t_1), \dots, \omega(t_n)) : t_1, \dots, t_n \in [0, T], \varphi \in C_{b,Lip}(\mathbb{R}^n), n \in \mathbb{N}\},$$

where  $C_{b,Lip}(\mathbb{R}^n)$  is the collection of bounded Lipschitz functions on  $\mathbb{R}^n$ .

We are given a function  $G : \mathbb{R} \mapsto \mathbb{R}$ , for  $0 \leq \underline{\sigma}^2 \leq \bar{\sigma}^2$ , by

$$G(a) := \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-).$$

For each  $\xi \in L_{ip}(\Omega_T)$  of the form

$$\xi(\omega) = \varphi(\omega(t_1), \omega(t_2), \dots, \omega(t_n)), \quad 0 = t_0 < t_1 < \dots < t_n = T,$$

we define the following conditional  $G$ -expectation

$$\mathbb{E}_t[\xi] := u_k(t, \omega(t); \omega(t_1), \dots, \omega(t_{k-1}))$$

for each  $t \in [t_{k-1}, t_k]$ ,  $k = 1, \dots, n$ . Here, for each  $k = 1, \dots, n$ ,  $u_k = u_k(t, x; x_1, \dots, x_{k-1})$  is a function of  $(t, x)$  parameterized by  $(x_1, \dots, x_{k-1}) \in \mathbb{R}^{k-1}$ , which is the solution of the following PDE ( $G$ -heat equation) defined on  $[t_{k-1}, t_k] \times \mathbb{R}$ :

$$\partial_t u_k + G(\partial_x^2 u_k) = 0$$

with terminal conditions

$$u_k(t_k, x; x_1, \dots, x_{k-1}) = u_{k+1}(t_k, x; x_1, \dots, x_{k-1}, x), \text{ for } k < n$$

and  $u_n(t_n, x; x_1, \dots, x_{n-1}) = \varphi(x_1, \dots, x_{n-1}, x)$ .

The  $G$ -expectation of  $\xi$  is defined by  $\mathbb{E}[\xi] = \mathbb{E}_0[\xi]$ . From this construction we obtain a natural norm  $\|\xi\|_{L_G^p} := \mathbb{E}[|\xi|^p]^{1/p}$ ,  $p \geq 1$ . The completion of  $L_{ip}(\Omega_T)$  under  $\|\cdot\|_{L_G^p}$  is a Banach space, denoted by  $L_G^p(\Omega_T)$ . The canonical process  $B_t(\omega) := \omega(t)$ ,  $t \geq 0$ , is called a  $G$ -Brownian motion in this sublinear expectation space  $(\Omega_T, L_G^1(\Omega_T), \mathbb{E})$ .

**Remark 2.1** For  $\varepsilon \in [0, \frac{\bar{\sigma}^2 - \underline{\sigma}^2}{2}]$ , set  $G_\varepsilon(a) = G(a) - \frac{\varepsilon}{2}|a|$ . Sometimes, we denote by  $\mathbb{E}_{G_\varepsilon}$  the  $G$ -expectation corresponds to the function  $G_\varepsilon$ .

**Definition 2.2** A process  $\{M_t\}$  with values in  $L_G^1(\Omega_T)$  is called a  $G$ -martingale if  $\mathbb{E}_s(M_t) = M_s$  for any  $s \leq t$ . If  $\{M_t\}$  and  $\{-M_t\}$  are both  $G$ -martingales, we call  $\{M_t\}$  a symmetric  $G$ -martingale.

**Theorem 2.3** ([1]) There exists a tight subset  $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ , the set of probability measures on  $(\Omega_T, \mathcal{B}(\Omega_T))$ , such that

$$\mathbb{E}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi] \text{ for all } \xi \in L_{ip}(\Omega_T).$$

$\mathcal{P}$  is called a set that represents  $\mathbb{E}$ .

**Remark 2.4** Let  $W_t$  be a one-dimensional standard Brownian motion in the probability space  $(\Omega, \mathcal{F}, P)$  and let  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  be the augmented filtration generated by  $(W_t)_{t \geq 0}$ . Denote by  $\mathcal{L}_{\mathbb{F}}^G$  the set of  $\mathbb{F}$ -adapted measurable processes with values in  $[\underline{\sigma}, \bar{\sigma}]$ . [1] showed that

$$\mathcal{P}_G := \{P_h | P_h := P \circ \left( \int_0^\cdot h_s dW_s \right)^{-1}, h \in \mathcal{L}_{\mathbb{F}}^G\}$$

is a set that represents  $\mathbb{E}$ .

**Definition 2.5** A function  $\eta(t, \omega) : [0, T] \times \Omega_T \rightarrow \mathbb{R}$  is called a step process if there exists a time partition  $\{t_i\}_{i=0}^n$  with  $0 = t_0 < t_1 < \dots < t_n = T$ , such that for each  $k = 0, 1, \dots, n-1$  and  $t \in (t_k, t_{k+1}]$

$$\eta(t, \omega) = \xi_{t_k} \in Lip(\Omega_{t_k}).$$

We denote by  $M^0(0, T)$  the collection of all step processes.

For each  $p \geq 1$ , we denote by  $M_G^p(0, T)$  the completion of the space  $M^0(0, T)$  under the norm

$$\|\eta\|_{M_G^p} := \left\{ \mathbb{E} \left[ \int_0^T |\eta_t|^p dt \right] \right\}^{1/p},$$

and by  $H_G^p(0, T)$  the completion of the space  $M^0(0, T)$  under the norm

$$\|\eta\|_{H_G^p} := \left[ \mathbb{E} \left[ \left\{ \int_0^T |\eta_t|^2 dt \right\}^{p/2} \right] \right]^{1/p}.$$

**Theorem 2.6** ([8], [9]) For  $\xi \in L_G^\beta(\Omega_T)$  with some  $\beta > 1$ ,  $X_t = \mathbb{E}_t(\xi)$ ,  $t \in [0, T]$  has the following decomposition:

$$X_t = \mathbb{E}[\xi] + \int_0^t Z_s dB_s + K_t, \text{ q.s.,}$$

where  $\{Z_t\} \in H_G^1(0, T)$  and  $\{K_t\}$  is a continuous non-increasing  $G$ -martingale. Furthermore, the above decomposition is unique and  $\{Z_t\} \in H_G^\alpha(0, T)$ ,  $K_T \in L_G^\alpha(\Omega_T)$  for any  $1 \leq \alpha < \beta$ .

### 3 Main results

In the sequel, we shall only consider the one-dimensional  $G$ -expectation space which is non-degenerate and really nonlinear, i.e.,  $\bar{\sigma} > \underline{\sigma} > 0$ .

Let  $W$  be a standard Brownian motion in the probability space  $(\Omega, \mathcal{F}, P)$  and assume that  $\mathbb{F} = (\mathcal{F}_t)$  is the augmented filtration generated by  $W$ .

An  $\mathbb{F}$ -adapted measurable process  $h$  is called an  $(m\text{-steps})$  self-dependent process if it has the following form:

$$h_t = \sum_{i=0}^{m-1} \xi_i 1_{\left[\frac{i}{m}, \frac{i+1}{m}\right]}(t) \quad (3.1)$$

where  $\xi_i = \varphi_i(\int_{\frac{i-1}{m}}^{\frac{i}{m}} h_s dW_s, \dots, \int_0^{\frac{1}{m}} h_s dW_s)$ ,  $\varphi_i \in C_{b, Lip}(R^i)$ . Clearly, an  $m$ -steps self-dependent process  $h$  can also be considered as a  $2^m m$ -steps self-dependent process for any  $n \geq 0$ .

**Lemma 3.1** (Lemma 4.2 in [11]) The collection of self-dependent processes bounded by two positive constants  $c, C$  ( $c \leq |h_s| \leq C$ ) is dense in the collection of  $\mathbb{F}$ -adapted measurable processes bounded by the same constants  $c, C$  under the norm

$$\|h\|_2 = [E(\int_0^1 |h_s|^2 ds)]^{1/2}.$$

Let  $B_t(\omega) = \omega_t$  be the canonical process on the space  $\Omega_T$ . For an  $\mathbb{F}$ -adapted measurable process  $h$ , set  $P_h = P \circ (\int_0^\cdot h_s dW_s)^{-1}$ , a probability on  $(\Omega_T, \mathcal{B}(\Omega_T))$ . For a process  $\{X_t\}$ , we denote by  $X_{[0, T]}^m$  the vector  $(X_T - X_{\frac{(m-1)T}{m}}, \dots, X_{\frac{T}{m}} - X_0)$ .

**Lemma 3.2** *Let  $h$  be an  $m$ -steps self-dependent process of form (3.1). We call a bounded  $\mathbb{F}$ -adapted measurable process  $\tilde{h}$  an  $m$ -perturbation of  $h$  if the following property holds:*

$$\int_{\frac{i}{m}}^{\frac{i+1}{m}} |\tilde{h}_s|^2 ds = \frac{1}{m} |\tilde{\xi}_i|^2 := \frac{1}{m} |\varphi_i(\int_{\frac{i-1}{m}}^{\frac{i}{m}} \tilde{h}_s dW_s, \dots, \int_0^{\frac{1}{m}} \tilde{h}_s dW_s)|^2, P\text{-a.s.}$$

*Then for any random variable of the form  $X = \psi(B_{[0,1]}^m)$  with  $\psi$  a bounded Lipschitz continuous function, we have*

$$E_{P_h}[X] = E_{P_{\tilde{h}}}[X].$$

**Proof.** Set  $\psi_1(x_{m-1}, \dots, x_1) := E[\psi(\varphi_{m-1}(x_{m-1}, \dots, x_1)(W_1 - W_{\frac{m-1}{m}}), x_{m-1}, \dots, x_1)]$ . On the one hand, we have

$$\begin{aligned} E_{P_h}[X] &= E[\psi(\int_{\frac{m-1}{m}}^1 h_s dW_s, \dots, \int_0^{\frac{1}{m}} h_s dW_s)] \\ &= E[E[\psi(\int_{\frac{m-1}{m}}^1 h_s dW_s, \dots, \int_0^{\frac{1}{m}} h_s dW_s) | \mathcal{F}_{\frac{m-1}{m}}]] \\ &= E[\psi_1(\int_{\frac{m-2}{m}}^{\frac{m-1}{m}} h_s dW_s, \dots, \int_0^{\frac{1}{m}} h_s dW_s)]. \end{aligned}$$

On the other hand, letting  $P_{\frac{m-1}{m}}^\omega$  be the regular conditional probability of  $P(\cdot | \mathcal{F}_{\frac{m-1}{m}})$ ,  $\int_{\frac{m-1}{m}}^1 \tilde{h}_s dW_s$  is normally distributed under  $P_{\frac{m-1}{m}}^\omega$  with mean 0 and variance  $\frac{1}{m} |\tilde{\xi}_{m-1}|^2(\omega)$  since  $\int_{\frac{i}{m}}^{\frac{i+1}{m}} |\tilde{h}_s|^2 ds = \frac{1}{m} |\tilde{\xi}_i|^2$ . So

$$\begin{aligned} E_{P_{\tilde{h}}}[X] &= E[\psi(\int_{\frac{m-1}{m}}^1 \tilde{h}_s dW_s, \dots, \int_0^{\frac{1}{m}} \tilde{h}_s dW_s)] \\ &= E[E[\psi(\int_{\frac{m-1}{m}}^1 \tilde{h}_s dW_s, \dots, \int_0^{\frac{1}{m}} \tilde{h}_s dW_s) | \mathcal{F}_{\frac{m-1}{m}}]] \\ &= E[\psi_1(\int_{\frac{m-2}{m}}^{\frac{m-1}{m}} \tilde{h}_s dW_s, \dots, \int_0^{\frac{1}{m}} \tilde{h}_s dW_s)]. \end{aligned}$$

Repeating the above arguments for  $m-1$  times, finally we can find a bounded Lipschitz continuous function  $\psi_{m-1}$  such that

$$E_{P_h}[X] = E_P[\psi_{m-1}(\int_0^{\frac{1}{m}} h_s dW_s)], \quad E_{P_{\tilde{h}}}[X] = E_P[\psi_{m-1}(\int_0^{\frac{1}{m}} \tilde{h}_s dW_s)].$$

Since  $\int_0^t |\tilde{h}_s|^2 ds = \frac{1}{m} \tilde{\xi}_0^2 = \frac{1}{m} \xi_0^2$ ,  $\int_0^{\frac{1}{m}} h_s dW_s$  and  $\int_0^{\frac{1}{m}} \tilde{h}_s dW_s$  are both normally distributed with mean 0 and variance  $\frac{1}{m} |\xi_0|^2$ . Hence, we have  $E_{P_h}[X] = E_{P_{\tilde{h}}}[X]$ .

□

**Theorem 3.3** *Let  $h$  be an  $m$ -steps self-dependent process. For  $n \geq 1$ , let  $h^n$  be a  $2^n m$ -perturbation of  $h$ . Assuming that  $(h^n)_{n \geq 1}$  are uniformly bounded, we have*

$$P_{h^n} \xrightarrow{w} P_h.$$

**Proof.** For any  $k \geq 1$  and any function  $\psi \in C_{b,Lip}(R^{2^k m})$ , by Lemma 3.2, we have, for  $n \geq k$ ,

$$E_{P_{h^n}}[\psi(B_{[0,1]}^{2^k m})] = E_{P_h}[\psi(B_{[0,1]}^{2^k m})].$$

In other words, we have

$$\lim_{n \rightarrow \infty} E_{P_{h^n}}[\psi(B_{[0,1]}^{2^k m})] = E_{P_h}[\psi(B_{[0,1]}^{2^k m})]$$

for any  $k \geq 1$  and any function  $\psi \in C_{b,Lip}(R^{2^k m})$ .

Since  $(h^n)_{n \geq 1}$  are uniformly bounded, we know that  $(P_{h^n})_{n \geq 1}$  are tight. Combing the above arguments, we conclude that

$$P_{h^n} \xrightarrow{w} P_h.$$

□

**Lemma 3.4** *Let  $K$  be a non-increasing  $G$ -martingale. Fix an  $\mathbb{F}$ -adapted measurable process  $h$  with  $\underline{\sigma} \leq |h| \leq \bar{\sigma}$ . Then for any  $s < t$  and any  $\varepsilon > 0$  there exists an  $\mathbb{F}$ -adapted measurable process  $\tilde{h}$  with  $\underline{\sigma} \leq |\tilde{h}| \leq \bar{\sigma}$  and  $\tilde{h}_r 1_{[0,s]}(r) = h_r 1_{[0,s]}(r)$  such that  $E_{P_{\tilde{h}}}[-(K_t - K_s)] < \varepsilon$ .*

**Proof.** Fix  $s < t$ ,  $\varepsilon > 0$  and an  $\mathbb{F}$ -adapted measurable process  $h$  with  $\underline{\sigma} \leq |h| \leq \bar{\sigma}$ . By Theorem 5.4 in [10], for the non-increasing  $G$ -martingale  $K_t$ , there exist  $\zeta \in M^0(0, T)$  such that

$$\mathbb{E}[\sup_{r \in [0,1]} |K_r - K_r(\zeta)|] < \frac{\varepsilon}{2},$$

where  $K_r(\zeta) = \int_0^r \zeta_u d\langle B \rangle_u - \int_0^r 2G(\zeta_u)du$ . We assume that  $\zeta$  is of the following form:

$$\zeta_u = \sum_{i=0}^{m-1} a_{t_i} 1_{[t_i, t_{i+1}]}(u),$$

where  $a_{t_i} = \phi_i(B_{t_i} - B_{t_{i-1}}, \dots, B_{t_1})$  with  $\phi_i \in C_{b,Lip}(R^i)$ . Without loss of generality, we assume  $s = t_i$  and  $t = t_{i+1}$ . Set  $\tilde{a}_{t_i} = \phi_i(\int_{t_{i-1}}^{t_i} h_u dW_u, \dots, \int_0^{t_1} h_u dW_u)$  and

$$\text{sign}_{\underline{\sigma}, \bar{\sigma}}(\tilde{a}_{t_i}) = \begin{cases} \bar{\sigma} & \text{if } \tilde{a}_{t_i} \geq 0; \\ \underline{\sigma} & \text{if } \tilde{a}_{t_i} < 0. \end{cases} \quad (3.2)$$

Let  $\tilde{h}_r = h_r$  for  $s \in [0, t_i]$  and let  $\tilde{h}_r = \text{sign}_{\underline{\sigma}, \bar{\sigma}}(\tilde{a}_{t_i})$  for  $r \in [t_i, t_{i+1}]$ . Then

$$\begin{aligned} E_{P_{\tilde{h}}}[K_t(\zeta) - K_s(\zeta)] &= E_{P_{\tilde{h}}}[a_{t_i}(\langle B \rangle_{t_{i+1}} - \langle B \rangle_{t_i}) - 2G(a_{t_i})(t_{i+1} - t_i)] \\ &= E[\tilde{a}_{t_i} \text{sign}_{\underline{\sigma}, \bar{\sigma}}(\tilde{a}_{t_i})^2(t_{i+1} - t_i) - 2G(\tilde{a}_{t_i})(t_{i+1} - t_i)] = 0. \end{aligned}$$

So

$$E_{P_{\tilde{h}}}[-(K_t - K_s)] \leq 2\mathbb{E}[\sup_{r \in [0,1]} |K_r - K_r(\zeta)|] < \varepsilon.$$

□

**Lemma 3.5** *Let  $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$  be a weakly compact set that represents  $\mathbb{E}$ :*

$$\mathbb{E}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi] \text{ for all } \xi \in L_{ip}(\Omega_T).$$

*Then, for  $\xi \in L_G^1(\Omega_T)$ ,  $(E_Q[\xi])_{Q \in \mathcal{P}}$  is continuous with respect to the weak convergence topology on  $\mathcal{M}_1(\Omega_T)$ .*

**Proof.** For  $\xi \in L_{ip}(\Omega_T)$ ,  $(E_Q[\xi])_{Q \in \mathcal{P}}$  is obviously continuous. By the definition of the space  $L_G^1(\Omega_T)$ , for  $\xi \in L_G^1(\Omega_T)$ ,  $(E_Q[\xi])_{Q \in \mathcal{P}}$  can be considered as the uniform limit of a sequence of continuous functions  $(E_Q[\xi^n])_{Q \in \mathcal{P}}$  with  $\xi^n \in L_{ip}(\Omega_T)$  and  $\mathbb{E}[|\xi^n - \xi|] \rightarrow 0$ . So we get the desired result.  $\square$

**Theorem 3.6** *Let  $K_t = \int_0^t \eta_s ds$  for some  $\eta \in M_G^1(0, T)$ . If  $K$  is a non-increasing  $G$ -martingale, we have  $K \equiv 0$ .*

**Proof.** Without loss of generality, we consider the case  $T = 1$ . Assume  $\mathbb{E}[-K_1] > 0$ . Then, by Lemma 3.5, there exists  $\varepsilon > 0$  such that  $\mathbb{E}_{G_\varepsilon}[-K_1] > 0$ . So, by Theorem 2.3 and Lemma 3.1, we can find a process  $h$  of form (3.1) with  $\underline{\sigma}^2 + \varepsilon \leq |h_s|^2 \leq \bar{\sigma}^2 - \varepsilon$  such that  $\delta := E_{P_h}[-K_1] > 0$ . Set  $\alpha = \frac{\varepsilon}{\bar{\sigma}^2 - \underline{\sigma}^2}$ . For  $k \geq 1$ , set

$$\delta_{k,\alpha}(s) = \sum_{i=0}^{k-1} (1_{\lfloor \frac{i}{k}, \frac{i+\alpha}{k} \rfloor}(s) - 1_{\lfloor \frac{i+\alpha}{k}, \frac{i+1}{k} \rfloor}(s)).$$

**Step 1.** For any  $n \geq 1$ , we can find a  $2^n m$ -perturbation  $h^n$  of  $h$  with  $\underline{\sigma} \leq |h_s^n| \leq \bar{\sigma}$  such that

$$E_{P_{h^n}} \left[ \int_0^1 \delta_{2^n m, \alpha}^+(s) \eta_s ds \right] > -\frac{\delta}{2} \alpha.$$

First, let us define  $h_s^n$  for  $s \in [0, \frac{1}{m}]$ .

Set  $\xi_0^n = \xi_0$ .

By Lemma 3.4, there exists an  $\mathbb{F}$ -adapted measurable process  $h^{n,1,1}$  with  $\underline{\sigma} \leq |h^{n,1,1}| \leq \bar{\sigma}$  such that  $E_{P_{h^{n,1,1}}}[-(K_{\frac{\alpha}{2^n m}} - K_0)] < \frac{\delta}{2^{n+1}m} \alpha$ .

Since  $\underline{\sigma}^2 + \varepsilon \leq |\xi_0^n|^2 \leq \bar{\sigma}^2 - \varepsilon$ , we have  $||h_s^{n,1,1}|^2 - |\xi_0^n|^2| \leq \bar{\sigma}^2 - \underline{\sigma}^2 - \varepsilon$ , by which we get  $\frac{\alpha}{1-\alpha} ||h_s^{n,1,1}|^2 - |\xi_0^n|^2| \leq \varepsilon$ , and consequently

$$\frac{2^n m}{1-\alpha} \int_0^{\frac{\alpha}{2^n m}} ||h_s^{n,1,1}|^2 - |\xi_0^n|^2| ds \leq \varepsilon.$$

So, noting  $\underline{\sigma}^2 + \varepsilon \leq |\xi_0^n|^2 \leq \bar{\sigma}^2 - \varepsilon$  again, we get

$$|\xi_0^n|^2 + \frac{2^n m}{1-\alpha} \int_0^{\frac{\alpha}{2^n m}} |\xi_0^n|^2 - |h_s^{n,1,1}|^2 ds = \frac{1}{1-\alpha} (|\xi_0^n|^2 - 2^n m \int_0^{\frac{\alpha}{2^n m}} |h_s^{n,1,1}|^2 ds) \in [\underline{\sigma}^2, \bar{\sigma}^2].$$

Set

$$h_s^n = \begin{cases} h_s^{n,1,1} & \text{for } s \in ]0, \frac{\alpha}{2^n m}]; \\ \sqrt{\frac{1}{1-\alpha} (|\xi_0^n|^2 - 2^n m \int_0^{\frac{\alpha}{2^n m}} |h_s^{n,1,1}|^2 ds)} & \text{for } s \in ]\frac{\alpha}{2^n m}, \frac{1}{2^n m}]. \end{cases} \quad (3.3)$$

It is easy to check that  $\int_0^{\frac{1}{2^n m}} |h_s^n|^2 ds = \frac{1}{2^n m} |\xi_0^n|^2$ .

Assume that we have defined  $h_s^n$  for all  $s \in [0, \frac{j}{2^n m}]$ ,  $0 \leq j \leq 2^n - 1$ . Then let us define  $h_s^n$  for  $s \in ]\frac{j}{2^n m}, \frac{j+1}{2^n m}]$ .

By Lemma 3.4, there exists an  $\mathbb{F}$ -adapted process  $h^{n,1,j+1}$  with  $\underline{\sigma} \leq |h^{n,1,j+1}| \leq \bar{\sigma}$  and  $h_r^{n,1,j+1} 1_{[0, \frac{j}{2^n m}]}(r) = h_r^n 1_{[0, \frac{j}{2^n m}]}(r)$  such that  $E_{P_{h^{n,1,j+1}}}[-(K_{\frac{j+\alpha}{2^n m}} - K_{\frac{j}{2^n m}})] < \frac{\delta}{2^{n+1}m} \alpha$ .

Set

$$h_s^n = \begin{cases} h_s^{n,1,j+1} & \text{for } s \in ]\frac{j}{2^n m}, \frac{j+\alpha}{2^n m}]; \\ \sqrt{\frac{1}{1-\alpha} (|\xi_0^n|^2 - 2^n m \int_0^{\frac{j+\alpha}{2^n m}} |h_s^{n,1,j+1}|^2 ds)} & \text{for } s \in ]\frac{j+\alpha}{2^n m}, \frac{j+1}{2^n m}]. \end{cases} \quad (3.4)$$

It is easily seen that  $\int_0^{\frac{j+1}{2^n m}} |h_s^n|^2 ds = \frac{1}{2^n m} |\xi_0^n|^2$  and  $|h_s^n|^2 \in [\underline{\sigma}^2, \bar{\sigma}^2]$ .



Assume that we have defined  $h_s^n$  for all  $s \in [0, \frac{i}{m}]$ ,  $0 \leq i \leq m-1$ .

Set  $\xi_i^n = \varphi_i(\int_{\frac{i-1}{m}}^{\frac{i}{m}} h_s^n dW_s, \dots, \int_0^{\frac{1}{m}} h_s^n dW_s)$ .

Then we can define the process  $h_s^n$  for  $s \in ]\frac{i}{m}, \frac{i+1}{m}]$  by repeating the above arguments with  $\xi_0^n$  replaced by  $\xi_i^n$ .

Clealy, the process  $h_s^n$  defined in this way is a  $2^n m$ -perturbation of  $h$ . Besides, we have

$$E_{P_{h^n}}[\int_0^1 \delta_{2^n m, \alpha}^+(s) \eta_s] = \sum_{j=0}^{2^n m-1} E_{P_{h^n}}[K_{\frac{j+\alpha}{2^n m}} - K_{\frac{j}{2^n m}}] > -\frac{\delta}{2}\alpha.$$

**Step 2.**  $\lim_{n \rightarrow \infty} \mathbb{E}[\int_0^1 \delta_{2^n m, \alpha}^-(s) \eta_s ds - (1-\alpha)K_1] = 0$ .

For  $\zeta \in M^0(0, 1)$ , the conclusion is obvious. As a functional of  $\zeta \in M_G^1(0, 1)$ ,

$$D_\alpha(\zeta) := \limsup_{n \rightarrow \infty} \mathbb{E}[\int_0^1 \delta_{2^n m, \alpha}^-(s) \zeta_s ds - (1-\alpha) \int_0^1 \zeta_s ds]$$

is continuous:  $|D_\alpha(\zeta) - D_\alpha(\varsigma)| \leq \|\zeta - \varsigma\|_{M_G^1}$ , for  $\zeta, \varsigma \in M_G^1(0, 1)$ , which implies the desired result.

**Step 3.**  $\lim_{n \rightarrow \infty} E_{P_{h^n}}[\int_0^1 \delta_{2^n m, \alpha}^-(s) \eta_s ds] = (1-\alpha)E_{P_h}[K_1] = -(1-\alpha)\delta$ .

Actually,

$$\begin{aligned} & |E_{P_{h^n}}[\int_0^1 \delta_{2^n m, \alpha}^-(s) \eta_s ds] - (1-\alpha)E_{P_h}[K_1]| \\ & \leq |E_{P_{h^n}}[\int_0^1 \delta_{2^n m, \alpha}^-(s) \eta_s ds] - (1-\alpha)E_{P_{h^n}}[K_1]| + (1-\alpha)|E_{P_{h^n}}[K_1] - E_{P_h}[K_1]| \\ & \leq \mathbb{E}[\int_0^1 \delta_{2^n m, \alpha}^-(s) \eta_s ds - (1-\alpha)K_1] + (1-\alpha)|E_{P_{h^n}}[K_1] - E_{P_h}[K_1]|. \end{aligned}$$

By Step 2 and Theorem 3.3, we get the desired result.

**Step 4.**  $\lim_{n \rightarrow \infty} \mathbb{E}[\int_0^1 (\delta_{2^n m, \alpha}^+(s) - \frac{\alpha}{1-\alpha} \delta_{2^n m, \alpha}^-(s)) \eta_s ds] = 0$ .

The proof follows immediately from Step 2. Actually, setting

$$d_\alpha(\zeta) := \limsup_{n \rightarrow \infty} \mathbb{E}[\int_0^1 (\delta_{2^n m, \alpha}^+(s) - \frac{\alpha}{1-\alpha} \delta_{2^n m, \alpha}^-(s)) \zeta_s ds], \quad \zeta \in M_G^1(0, 1),$$

it is easily seen that  $(1-\alpha)d_\alpha(\zeta) = D_\alpha(\zeta)$ .

Combing the above arguments, we get

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \mathbb{E}[\int_0^1 (\delta_{2^n m, \alpha}^+(s) - \frac{\alpha}{1-\alpha} \delta_{2^n m, \alpha}^-(s)) \eta_s ds] \\ &\geq \limsup_{n \rightarrow \infty} E_{P_{h^n}}[\int_0^1 (\delta_{2^n m, \alpha}^+(s) - \frac{\alpha}{1-\alpha} \delta_{2^n m, \alpha}^-(s)) \eta_s ds] \\ &\geq \limsup_{n \rightarrow \infty} E_{P_{h^n}}[\int_0^1 \delta_{2^n m, \alpha}^+(s) \eta_s] - \frac{\alpha}{1-\alpha} \lim_{n \rightarrow \infty} E_{P_{h^n}}[\int_0^1 \delta_{2^n m, \alpha}^-(s) \eta_s ds] \\ &\geq -\frac{\delta}{2}\alpha + \frac{\alpha}{1-\alpha} \times (1-\alpha)\delta = \frac{\delta}{2}\alpha > 0, \end{aligned}$$

which is a contradiction. The last inequality follows from Step 1 and Step 3.  $\square$

**Corollary 3.7** Let  $K_t = \int_0^t \eta_s d\langle B \rangle_s$  for some  $\eta \in M_G^1(0, T)$ . If  $K$  is a non-increasing  $G$ -martingale, we have  $K \equiv 0$ .

**Proof.** Set  $L_t = \int_0^t \eta_s ds$ . Assume that  $K$  is a non-increasing  $G$ -martingale. Then we have

$$0 \geq \mathbb{E}_s[L_t - L_s] \geq \frac{1}{\underline{\sigma}^2} \mathbb{E}_s[K_t - K_s] = 0.$$

So  $L$  is a non-increasing  $G$ -martingale. By Theorem 3.6, we get  $L \equiv 0$ , and consequently,  $K \equiv 0$ .  $\square$

As an application of Theorem 3.6, we shall prove the uniqueness of the decomposition for generalized  $G$ -Itô processes.

**Definition 3.8** *A process of the following form is called a generalized  $G$ -Itô process:*

$$u = u_0 + \int_0^t \eta_s ds + \int_0^t \zeta_s dB_s + K_t,$$

where  $\eta \in M_G^1(0, T)$ ,  $\zeta \in H_G^1(0, T)$  and  $K$  is a non-increasing  $G$ -martingale.

**Remark 3.9** *A  $G$ -Itô process*

$$u = u_0 + \int_0^t \tau_s ds + \int_0^t \zeta_s dB_s + \int_0^t \frac{1}{2} \gamma_s d\langle B \rangle_s, \quad \tau, \gamma \in M_G^1(0, T), \quad \zeta \in H_G^1(0, T),$$

can be rewritten as

$$u = u_0 + \int_0^t (\tau_s + G(\gamma_s)) ds + \int_0^t \zeta_s dB_s + K_t,$$

where  $K_t = \int_0^t \frac{1}{2} \gamma_s d\langle B \rangle_s - \int_0^t G(\gamma_s) ds$ , which, as is known, is a non-increasing  $G$ -martingale. So a  $G$ -Itô process is a generalized  $G$ -Itô process.

By Corollary 3.5 in Song (2012) we conclude that the decomposition for  $G$ -Itô processes is unique. The next result shows the uniqueness of the decomposition for generalized  $G$ -Itô processes.

**Theorem 3.10** *Assume  $\int_0^t \zeta_s dB_s + \int_0^t \eta_s ds + K_t = L_t$ , where  $\zeta \in H_G^1(0, T)$ ,  $\eta \in M_G^1(0, T)$ , and  $K_t, L_t$  are non-increasing  $G$ -martingales. Then we have  $\int_0^t \zeta_s dB_s \equiv 0$ ,  $\int_0^t \eta_s ds \equiv 0$  and  $K_t = L_t$ .*

**Proof.** By the uniqueness for the decomposition for (classical) continuous semimartingales, we get  $\int_0^t \zeta_s dB_s \equiv 0$ . Assume  $\int_0^t \eta_s ds + K_t = L_t$ . Since  $L_t$  is non-increasing,  $\tilde{L}_t := \int_0^t \eta_s^+ ds + K_t$  is also non-increasing, which implies that  $-\int_s^t \eta_r^+ dr \geq K_t - K_s$  for any  $s < t$ . Noting that  $0 \geq \mathbb{E}_s[-\int_s^t \eta_r^+ dr] \geq \mathbb{E}_s[K_t - K_s] = 0$  since  $K$  is a  $G$ -martingale, we conclude that  $-\int_0^t \eta_s^+ ds$  is also a  $G$ -martingale, which implies, by Theorem 3.6, that  $\int_0^t \eta_s^+ ds = 0$ . By the same arguments, we have  $\int_0^t \eta_s^- ds = 0$ .  $\square$

**Corollary 3.11** *Assume that  $K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t \zeta_s ds$ ,  $\eta, \zeta \in M_G^1(0, T)$ , is a non-increasing  $G$ -martingale. Then we have  $\zeta \equiv 2G(\eta)$ .*

**Proof.** Since  $L_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds = K_t + \int_0^t 2G(\eta_s) - \zeta_s ds$  is a non-increasing  $G$ -martingale, by Theorem 3.10, we get  $\zeta \equiv 2G(\eta)$ .  $\square$

**Remark 3.12** *Theorem 3.10 turns out to be a very strong conclusion. Many important results in the context of  $G$ -expectation theory can be considered as its immediate corollaries.*

1) Theorem 3.6 in [12]: A process  $A_t = \int_0^t \eta_s d\langle B \rangle_s$ ,  $\eta \in M_G^p(0, T)$  for some  $p > 1$ , has stationary and independent increments if and only if  $A_t = c\langle B \rangle_t$  for some constant  $c \in \mathbb{R}$ .

**Proof.** We only prove the “only if” part.

Assume that  $A$  is a process with stationary and independent increments. Then there exists a constant  $\lambda \in \mathbb{R}$  such that  $\mathbb{E}[A_t] = \lambda t$  and  $L_t := A_t - \lambda t$  is a non-increasing  $G$ -martingale. So we conclude by Corollary 3.11 that  $\lambda = 2G(\eta_s)$ , which implies the desired conclusion.  $\square$

2) Corollary 3.5 in [11]: If  $\int_0^t \eta_s d\langle B \rangle_s = \int_0^t \zeta ds$  for some  $\eta, \zeta \in M_G^1(0, T)$ , we have  $\eta = \zeta \equiv 0$ .

**Proof.** By the assumption, we have

$$\int_0^t \eta_s d\langle B \rangle_s =: K_t + \int_0^t 2G(\eta_s) ds = \int_0^t \zeta ds.$$

By Theorem 3.10, we get  $K_t = \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds \equiv 0$ . For any  $\varepsilon \in [0, \frac{\bar{\sigma}^2 - \sigma^2}{2}]$  we have

$$0 = \mathbb{E}[-K_T] \geq \mathbb{E}_{G_\varepsilon}[-K_T] \geq \varepsilon \mathbb{E}_{G_\varepsilon} \left[ \int_0^T |\eta_s| ds \right],$$

which implies  $\eta \equiv 0$ , and consequently,  $\zeta \equiv 0$ .  $\square$

## 4 Application: characterization of $G$ -Sobolev space $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$

Peng and Song (2015) introduced the notion of  $G$ -Sobolev spaces, in which they defined solutions to a certain type of path dependent PDEs.

Their definitions of  $G$ -Sobolev spaces started from the following spaces of smooth functions of paths.

**Definition 4.1** A function  $\xi : \Omega_T \rightarrow \mathbb{R}$  is called a cylinder function of paths on  $[0, T]$  if it can be represented by

$$\xi(\omega) = \varphi(\omega(t_1), \dots, \omega(t_n)), \omega \in \Omega_T,$$

for some  $0 = t_0 < \dots < t_n = T$ , where  $\varphi : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$  is a  $C^\infty$ -function with at most polynomial growth. We denote by  $C^\infty(\Omega_T)$  the collection of all cylinder functions of paths on  $[0, T]$ .

**Definition 4.2** A function  $u(t, \omega) : [0, T] \times \Omega_T \rightarrow \mathbb{R}$  is called a cylinder path process if there exists a time partition  $0 = t_0 < \dots < t_n = T$ , such that for each  $k = 0, 1, \dots, n-1$  and  $t \in (t_k, t_{k+1}]$ ,

$$u(t, \omega) = u_k(t, \omega(t); \omega(t_1), \dots, \omega(t_k)).$$

Here for each  $k$ , the function  $u_k : [t_k, t_{k+1}] \times (\mathbb{R}^d)^{k+1} \rightarrow \mathbb{R}$  is a  $C^\infty$ -function with

$$u_k(t_k, x; x_1, \dots, x_{k-1}, x) = u_{k-1}(t_k, x; x_1, \dots, x_{k-1})$$

such that, all derivatives of  $u_k$  have at most polynomial growth. We denote by  $\mathcal{C}^\infty(0, T)$  the collection of all cylinder path processes.

For a function  $u \in \mathcal{C}^\infty(0, T)$ , set, for  $t \in [t_k, t_{k+1}]$ ,

$$\mathcal{D}_t u(t, \omega) := \partial_t u_k(t, x; x_1, \dots, x_k) \big|_{x=\omega(t), x_1=\omega(t_1), \dots, x_k=\omega(t_k)}, \quad (4.1)$$

$$\mathcal{D}_x u(t, \omega) := \partial_x u_k(t, x; x_1, \dots, x_k) \big|_{x=\omega(t), x_1=\omega(t_1), \dots, x_k=\omega(t_k)}, \quad (4.2)$$

$$\mathcal{D}_x^2 u(t, \omega) := \partial_x^2 u_k(t, x; x_1, \dots, x_k) \big|_{x=\omega(t), x_1=\omega(t_1), \dots, x_k=\omega(t_k)}. \quad (4.3)$$

Set

$$\mathcal{A}_G u(t, \omega) := \mathcal{D}_t u(t, \omega) + G(\mathcal{D}_x^2 u(t, \omega)).$$

#### 4.1 $G$ -Sobolev spaces $W_G^{1,2;p}(0, T)$ introduced in [7]

**Definition 4.3** 1) For  $u \in \mathcal{C}^\infty(0, T)$ , we set

$$\|u\|_{S_G^p}^p = \mathbb{E}[\sup_{s \in [0, T]} |u(s, \omega)|^p].$$

We denote by  $S_G^p(0, T)$  the completion of  $\mathcal{C}^\infty(0, T)$  w.r.t. the norm  $\|\cdot\|_{S_G^p}$ .

2) For  $u \in \mathcal{C}^\infty(0, T)$ , we set

$$\|u\|_{W_G^{1,2;p}}^p = \mathbb{E}[\sup_{s \in [0, T]} |u_s|^p + \int_0^T (|\mathcal{D}_s u_s|^p + |\mathcal{D}_x^2 u_s|^p) ds + (\int_0^T |\mathcal{D}_x u_s|^2 ds)^{p/2}].$$

Denote by  $W_G^{1,2;p}(0, T)$  the completion of  $\mathcal{C}^\infty(0, T)$  w.r.t. the norm  $\|\cdot\|_{W_G^{1,2;p}}$ .

Sometimes, we shall abuse notations by writing  $u(t, \omega)$  as  $u_t$  for simplicity.

By Corollary 3.5 in Song (2012) we conclude that the decomposition for  $G$ -Itô processes is unique: letting  $\tau, \gamma \in M_G^1(0, T)$ ,  $\zeta \in H_G^1(0, T)$ , then

$$\int_0^t \tau_s ds + \int_0^t \zeta_s dB_s + \int_0^t \frac{1}{2} \gamma_s d\langle B \rangle_s = 0$$

implies that  $\tau = \gamma \equiv 0$  and  $\zeta \equiv 0$ .

From this it is easily seen that the norm  $\|\cdot\|_{W_G^{1,2;p}}$  is closable in the space  $S_G^p(0, T)$ : Let  $u^n \in \mathcal{C}^\infty(0, T)$  be a Cauchy sequence w.r.t. the norm  $\|\cdot\|_{W_G^{1,2;p}}$ . If  $\|u^n\|_{S_G^p} \rightarrow 0$ , we have  $\|u^n\|_{W_G^{1,2;p}} \rightarrow 0$ .

**Remark 4.4** The closability of the norm  $\|\cdot\|_{W_G^{1,2;p}}$ , which follows from the uniqueness of the decomposition for  $G$ -Itô processes, is the key point to extend the definition of the operators  $\mathcal{D}_t$ ,  $\mathcal{D}_x$ ,  $\mathcal{D}_x^2$  to the space  $W_G^{1,2;p}(0, T)$ . Precisely, unless the norm  $\|\cdot\|_{W_G^{1,2;p}}$  is closable, a process  $u \in S_G^p(0, T)$  may correspond to two different elements in  $W_G^{1,2;p}(0, T)$ , which can be represented as:  $(u, \tau, \zeta, \gamma)$  and  $(u, \tilde{\tau}, \tilde{\zeta}, \tilde{\gamma})$ ,  $\tau, \tilde{\tau}, \gamma, \tilde{\gamma} \in M_G^p(0, T)$ ,  $\zeta, \tilde{\zeta} \in H_G^p(0, T)$ . For this case, we could not well-define the derivatives for  $u$ .

So  $W_G^{1,2;p}(0, T)$  can be considered as a subspace of  $S_G^p(0, T)$ , and the derivative operators  $\mathcal{D}_t$ ,  $\mathcal{D}_x^2$  (resp.  $\mathcal{D}_x$ ), can all be extended as continuous linear operators from  $W_G^{1,2;p}(0, T)$  to  $M_G^p(0, T)$  (resp. to  $H_G^p(0, T)$ ).

**Theorem 4.5** (Theorem 4.5 in [7]) Assume  $u \in S_G^p(0, T)$ . Then the following two conditions are equivalent:

- (i)  $u \in W_G^{1,2;p}(0, T)$ ;
- (ii) There exists  $u_0 \in \mathbb{R}$ ,  $\zeta, w \in M_G^p(0, T)$  and  $v \in H_G^p(0, T)$  such that

$$u_t = u_0 + \int_0^t \zeta_s ds + \int_0^t v_s dB_s + \frac{1}{2} \int_0^t w_s d\langle B \rangle_s. \quad (4.4)$$

Moreover, we have

$$\mathcal{D}_t u_t = \zeta_t, \quad \mathcal{D}_x u_t = v_t, \quad \mathcal{D}_x^2 u_t = w_t.$$

In [7], the authors defined  $W_G^{1,2;p}(0, T)$ -solutions to path dependent PDEs and established one-one correspondence between backward SDEs under  $G$ -expectation and a certain type of path dependent PDEs.

**Backward SDEs:** to find  $Y \in S_G^p(0, T)$ ,  $Z \in H_G^p(0, T)$ ,  $\eta \in M_G^p(0, T)$  such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \eta_s) ds - \int_t^T Z_s dB_s - \left( \int_t^T \frac{1}{2} \eta_s d\langle B \rangle_s - \int_t^T G(\eta_s) ds \right), \quad (4.5)$$

where  $f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d) \mapsto \mathbb{R}$  is a given function and  $\xi \in L_G^p(\Omega_T)$  is a given random variable.

**Path Dependent PDEs:** to find a path-dependent  $u \in W_G^{1,2;p}(0, T)$  such that

$$\begin{aligned} \mathcal{D}_t u + G(\mathcal{D}_x^2 u) + f(t, u, \mathcal{D}_x u, \mathcal{D}_x^2 u) &= 0, \quad t \in [0, T], \\ u_T &= \xi. \end{aligned} \quad (4.6)$$

We assume that  $f(t, \omega, Y_t, Z_t, \eta_t) \in M_G^p(0, T)$  for any  $(Y, Z, \eta) \in S_G^p(0, T) \times H_G^p(0, T) \times M_G^p(0, T)$ .

**Theorem 4.6** (*Theorem 4.9 in [7]*) *Let  $(Y, Z, \eta)$  be a solution to the backward SDE (4.5). Then we have  $u_t := Y_t \in W_G^{1,2;p}(0, T)$  with  $\mathcal{D}_x u_t = Z_t$  and  $\mathcal{D}_x^2 u_t = \eta_t$ .*

*Moreover, Given a  $u \in W_G^{1,2;p}(0, T)$ , the following conditions are equivalent:*

- (i)  *$(u, \mathcal{D}_x u, \mathcal{D}_x^2 u)$  is a solution to the backward SDE (4.5);*
- (ii)  *$u$  is a  $W_G^{1,2;p}$ -solution to the path dependent PDE (4.6).*

## 4.2 Characterization of $G$ -Sobolev space $W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0, T)$

Now we consider a special case of the path dependent PDE (4.6):  $f$  is independent of  $\mathcal{D}_x^2 u$ .

$$\begin{aligned} \mathcal{D}_t u + G(\mathcal{D}_x^2 u) + f(t, u, \mathcal{D}_x u) &= 0, \quad t \in [0, T], \\ u_T &= \xi. \end{aligned} \quad (4.7)$$

Let  $u \in W_G^{1,2;p}(0, T)$  be a solution to the path dependent PDE (4.7). By Theorem 4.6, the processes

$$Y_t := u_t, \quad Z_t := \mathcal{D}_x u_t, \quad K_t := \frac{1}{2} \int_0^t \mathcal{D}_x^2 u_s d\langle B \rangle_s - \int_0^t G(\mathcal{D}_x^2 u_s) ds$$

satisfy the following backward SDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t), \quad (4.8)$$

which is nothing but the backward SDEs driven by  $G$ -Brownian motion ( $G$ -BSDE) studied in [2].

On the contrary, letting  $(Y, Z, K)$  be a solution of backward SDE (4.8) considered in [2], notice that, although we have many interesting examples, but it is still a very interesting and challenging problem to give reasonable conditions on  $\xi$  and  $f$  such that  $Y$  lies in the Sobolev space  $W_G^{1,2;p}(0, T)$ . Even so, we still think  $u = Y$  is a reasonable candidate of the solution to Equ. (4.7).

In [7], the authors formulated  $u = Y$  as the unique solution to Equ. (4.7) in a first order Sobolev space  $W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0, T)$ . In this section, we shall refine the definition of the  $G$ -Sobolev space  $W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0, T)$ . The main idea is, just like the liner case, to integrate  $\mathcal{A}_G u = \mathcal{D}_t u + G(\mathcal{D}_x^2 u)$  as one operator, which reduces the regularity requirement for the solutions. To well define the derivative  $\mathcal{A}_G u$  for  $u \in W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0, T)$ , the uniqueness of the decomposition for generalized  $G$ -Itô processes plays a crucial role.

#### 4.2.1 Definition of the $G$ -Sobolev space $W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0,T)$

For  $u, v \in \mathcal{C}^\infty(0, T)$ , set

$$d_{G,p}^p(u, v) = \mathbb{E} \left[ \sup_{s \in [0, T]} |u_s - v_s|^p + \left( \int_0^T |\mathcal{D}_x(u_s - v_s)|^2 ds \right)^{\frac{p}{2}} + \int_0^T |\mathcal{A}_G u_s - \mathcal{A}_G v_s|^p ds \right].$$

By the uniqueness of the decomposition for generalized  $G$ -Itô processes we obtain the closability of the metric  $d_{G,p}$ . As is stated in Remark 4.4, closability of the metric  $d_{G,p}$  is the key point to well define the operators  $\mathcal{A}_G$ ,  $\mathcal{D}_x$ .

**Proposition 4.7** *The metric  $d_{G,p}$  is closable in the space  $S_G^p(0, T)$ : Let  $u^n, v^n \in \mathcal{C}^\infty(0, T)$  be two Cauchy sequences w.r.t. the metric  $d_{G,p}$ . If  $\|u^n - v^n\|_{S_G^p} \rightarrow 0$ , we have  $d_{G,p}(u^n, v^n) \rightarrow 0$ .*

**Proof.** For  $u^n \in \mathcal{C}^\infty(0, T)$ , by Itô's formula, we have

$$\begin{aligned} u^n(t, \omega) &= u^n(0, \omega) + \int_0^t \mathcal{D}_s u^n(s, \omega) ds + \int_0^t \mathcal{D}_x u^n(s, \omega) dB_s + \frac{1}{2} \int_0^t \mathcal{D}_x^2 u^n(s, \omega) d\langle B \rangle_s \\ &= u^n(0, \omega) + \int_0^t \mathcal{A}_G u^n(s, \omega) ds + \int_0^t \mathcal{D}_x u^n(s, \omega) dB_s + K_t^n, \end{aligned}$$

where  $K_t^n := \frac{1}{2} \int_0^t \mathcal{D}_x^2 u^n(s, \omega) d\langle B \rangle_s - \int_0^t G(\mathcal{D}_x^2 u^n(s, \omega)) ds$  is a non-increasing  $G$ -martingale. If  $(u^n)_n$  is a Cauchy sequence w.r.t. the metric  $d_{G,p}$ , there will be processes  $u \in S_G^p(0, T)$ ,  $\eta \in M_G^p(0, T)$ ,  $\zeta \in H_G^p(0, T)$  such that

$$\|u^n - u\|_{S_G^p} + \|\mathcal{A}_G u^n - \eta\|_{M_G^p} + \|\mathcal{D}_x u^n - \zeta\|_{H_G^p} \rightarrow 0.$$

Set  $K_t = u_t - u_0 - \int_0^t \eta_s ds - \int_0^t \zeta_s dB_s$ . It is easily seen that  $\|K^n - K\|_{S_G^p} \rightarrow 0$ . So  $K$  is a non-increasing  $G$ -martingale and  $u_t = u_0 + \int_0^t \eta_s ds + \int_0^t \zeta_s dB_s + K_t$  is a generalized  $G$ -Itô process. Assuming  $(v^n)_n$  is a Cauchy sequence w.r.t. the metric  $d_{G,p}$ , similarly, there exists a generalized  $G$ -Itô process  $\tilde{u}_t = \tilde{u}_0 + \int_0^t \tilde{\eta}_s ds + \int_0^t \tilde{\zeta}_s dB_s + \tilde{K}_t$  such that

$$\|v^n - \tilde{u}\|_{S_G^p} + \|\mathcal{A}_G v^n - \tilde{\eta}\|_{M_G^p} + \|\mathcal{D}_x v^n - \tilde{\zeta}\|_{H_G^p} \rightarrow 0.$$

If  $\|u^n - v^n\|_{S_G^p} \rightarrow 0$ , we get  $u = \tilde{u}$ . By the uniqueness of the decomposition for generalized  $G$ -Itô processes, we get  $\eta = \tilde{\eta}$  and  $\zeta = \tilde{\zeta}$ , which implies  $d_{G,p}(u^n, v^n) \rightarrow 0$ .

□

Denote by  $W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0, T)$  the closure of  $\mathcal{C}^\infty(0, T)$  w.r.t. the metric  $d_{G,p}$  in  $S_G^p(0, T)$ . Now the operators  $\mathcal{A}_G$ ,  $\mathcal{D}_x$  can be continuously extended to the space  $W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0, T)$ .

**Proposition 4.8** *Assume  $u \in S_G^p(0, T)$ . Then the following two conditions are equivalent:*

- (i)  $u \in W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0, T)$ ;
- (ii) There exists  $\eta \in M_G^p(0, T)$  and  $\zeta \in H_G^p(0, T)$  such that

$$u(t, \omega) = u_0 + \int_0^t \eta(s, \omega) ds + \int_0^t \zeta(s, \omega) dB_s$$

is a non-increasing  $G$ -martingale, namely,  $u$  is a generalized  $G$ -Itô process.

Moreover, we have  $\mathcal{A}_G u = \eta$  and  $\mathcal{D}_x u = \zeta$ .

**Proof.** (i)  $\implies$  (ii) is obvious.

(ii)  $\implies$  (i). Let  $u$  be a generalized  $G$ -Itô process. By Theorem 5.4 in [10], it suffices to prove the claim for  $u$  of the following form:

$$u_t = u_0 + \int_0^t \eta_s ds + \int_0^t \zeta_s dB_s + \frac{1}{2} \int_0^t w_s d\langle B \rangle_s - \int_0^t G(w_s) ds,$$

where  $\eta, \zeta, w$  are smooth step processes, namely,  $\eta, \zeta, w \in M^0(0, T)$  and  $\eta_t, \zeta_t, w_t \in C^\infty(\Omega_t)$  (Def. 4.1). Set  $t_k^n = \frac{kT}{2^n}$  and  $Q_t^n := \sum_{k=0}^{2^n-1} (B_{t_{k+1}^n \wedge t} - B_{t_k^n \wedge t})^2 = \int_0^t \lambda_s^n dB_s + \langle B \rangle_t$ , where  $\lambda_t^n = \sum_{k=0}^{2^n-1} 2(B_t - B_{t_k^n})1_{[t_k^n, t_{k+1}^n]}(t)$ . Choose a sequence of smooth step processes  $\alpha_s^n$  such that  $\mathbb{E}[\int_0^T |\alpha_s^n - G(w_s)|^p ds] \rightarrow 0$ . Set

$$u_t^n := u_0 + \int_0^t \eta_s ds + \int_0^t \zeta_s dB_s + \frac{1}{2} \int_0^t w_s dQ_s^n - \int_0^t \alpha_s^n ds \quad (4.9)$$

$$= u_0 + \int_0^t (\eta_s - \alpha_s^n) ds + \int_0^t (\zeta_s + \frac{1}{2} w_s \lambda_s^n) dB_s + \int_0^t \frac{1}{2} w_s d\langle B \rangle_s. \quad (4.10)$$

It is easily seen, by (4.9), that  $u^n$  belongs to  $C^\infty(0, T)$ . By the uniqueness of the decomposition for  $G$ -Itô processes and (4.10) we know that

$$\mathcal{D}_t u_t^n = \eta_t - \alpha_t^n, \quad \mathcal{D}_x u_t^n = \zeta_t + \frac{1}{2} w_t \lambda_t^n, \quad \mathcal{D}_x^2 u_t^n = w_t.$$

So  $\mathcal{A}_G u_t^n = \eta_t - \alpha_t^n + G(w_t)$ . It is easy to show that  $\mathcal{A}_G u^n \xrightarrow{M_G^p} \eta$ ,  $\mathcal{D}_x u^n \xrightarrow{H_G^p} \zeta$ ,  $u^n \xrightarrow{S_G^p} u$ . So  $u \in W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$  with  $\mathcal{A}_G u = \eta$  and  $\mathcal{D}_x u = \zeta$ .  $\square$

#### 4.2.2 Fully nonlinear path dependent PDEs

Let us define the  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}$ -solution to the path dependent PDE (4.7) : to find  $u \in W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$  such that

$$\begin{aligned} \mathcal{A}_G u + f(t, u, \mathcal{D}_x u) &= 0, \quad t \in [0, T], \\ u_T &= \xi. \end{aligned} \quad (4.11)$$

Now we can interpret backward SDEs driven by  $G$ -Brownian motion as “path dependent” PDEs.

We assume that  $f(t, \omega, Y_t, Z_t) \in M_G^p(0, T)$  for any  $(Y, Z) \in S_G^p(0, T) \times H_G^p(0, T)$ .

**Theorem 4.9** *Let  $(Y, Z)$  be a solution to the backward SDE (4.8) (see Def. 5.1 and Rem. 5.3). Then we have  $u_t := Y_t \in W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$  with  $\mathcal{D}_x u_t = Z_t$ .*

*Moreover, Given a  $u \in W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$ , the following conditions are equivalent:*

- (i)  $(u, \mathcal{D}_x u)$  is a solution to the backward SDE (4.8);
- (ii)  $u$  is a  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}(0, T)$ -solution to the path dependent PDE (4.11).

**Proof.** The proof follows immediately from Proposition 4.8 and the definitions of the solutions to the backward SDE (4.8) and the path dependent PDE (4.11).  $\square$

Assume that the function  $g(t, \omega, y, z) : [0, T] \times \Omega_T \times R \times R \rightarrow R$  satisfies the following assumptions: there exists some  $\beta > 1$  such that

**(H1)** for any  $y, z$ ,  $g(t, \omega, y, z) \in M_G^\beta(0, T)$ ;

**(H2)**  $|g(t, \omega, y, z) - g(t, \omega, y', z')| \leq L(|y - y'| + |z - z'|)$  for some constant  $L > 0$ .

**Corollary 4.10** Assume  $\xi \in L_G^\beta(\Omega_T)$  and  $g$  satisfies (H1) and (H2) for some  $\beta > 1$ . Then, for each  $p \in (1, \beta)$ , the path dependent PDE (4.7) has a unique  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}$ -solution  $u$ .

In particular, the martingale  $u(t, \omega) := \mathbb{E}_t[\xi](\omega)$  is the unique  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}$ -solution of the path dependent  $G$ -heat equation

$$\mathcal{D}_t u + G(\mathcal{D}_x^2 u) = 0, \quad u_T = \xi.$$

**Proof.** The uniqueness is straightforward from Theorem 4.9 and Theorem 5.2.

We now prove the existence. By Theorem 5.2 we know that the backward SDE (4.8) has a solution  $(Y, Z)$ . By the assumption (H1) and (H2), we conclude  $g(t, \omega, Y_t(\omega), Z_t(\omega)) \in M_G^p(0, T)$ . So we get the existence result from Theorem 4.9.

By the  $G$ -martingale decomposition theorem,  $u \in S_G^p(0, T)$  is a  $G$ -martingale if and only if  $u$  is a solution of backward SDE (4.8) with  $f = 0$ . So  $u(t, \omega) := \mathbb{E}_t[\xi](\omega)$  is the unique  $W_{\mathcal{A}_G}^{\frac{1}{2}, 1; p}$ -solution of the path dependent  $G$ -heat equation.  $\square$

## 5 Appendix: Backward SDEs driven by $G$ -Brownian motion

In [2] the authors studied the backward stochastic differential equations driven by a  $G$ -Brownian motion  $(B_t)_{t \geq 0}$  in the following form:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t). \quad (5.1)$$

where  $K$  is a non-increasing  $G$ -martingale.

The main result in [2] is the existence and uniqueness of a solution  $(Y, Z, K)$  for equation (5.1) in the  $G$ -framework under the following assumption: there exists some  $\beta > 1$  such that (H1) and (H2) are satisfied.

**Definition 5.1** Let  $\xi \in L_G^\beta(\Omega_T)$  and  $g$  satisfy (H1) and (H2) for some  $\beta > 1$ . A triplet of processes  $(Y, Z, K)$  is called a solution of equation (5.1) if for some  $1 < \alpha \leq \beta$  the following properties hold:

(a)  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T)$ ,  $K$  is a non-increasing  $G$ -martingale with  $K_0 = 0$  and  $K_T \in L_G^\alpha(\Omega_T)$ ;

(b)  $Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t)$ .

The main result in [2] is the following theorem:

**Theorem 5.2** Assume that  $\xi \in L_G^\beta(\Omega_T)$  and  $f$  satisfies (H1) and (H2) for some  $\beta > 1$ . Then equation (5.1) has a unique solution  $(Y, Z, K)$ . Moreover, for any  $1 < \alpha < \beta$  we have  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T)$  and  $K_T \in L_G^\alpha(\Omega_T)$ .

**Remark 5.3** Equivalently, we say a pair of processes  $(Y, Z)$  is a solution of equation (5.1) if for some  $1 < \alpha \leq \beta$  the following properties hold:

(a)  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T)$ ;

(b)  $Y_T = \xi$  and  $K_t := Y_t + \int_0^t g(s, Y_s, Z_s) ds - \int_0^t Z_s dB_s$  is a non-increasing  $G$ -martingale.



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